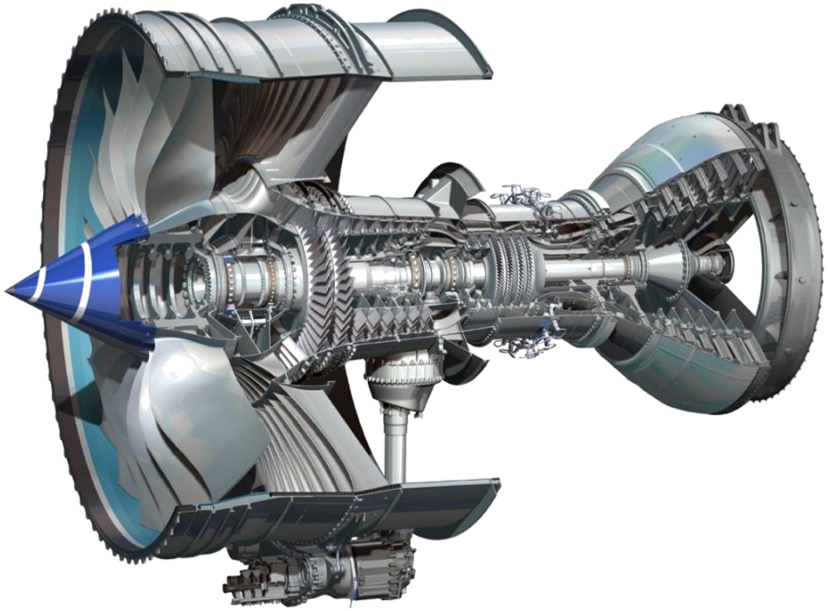
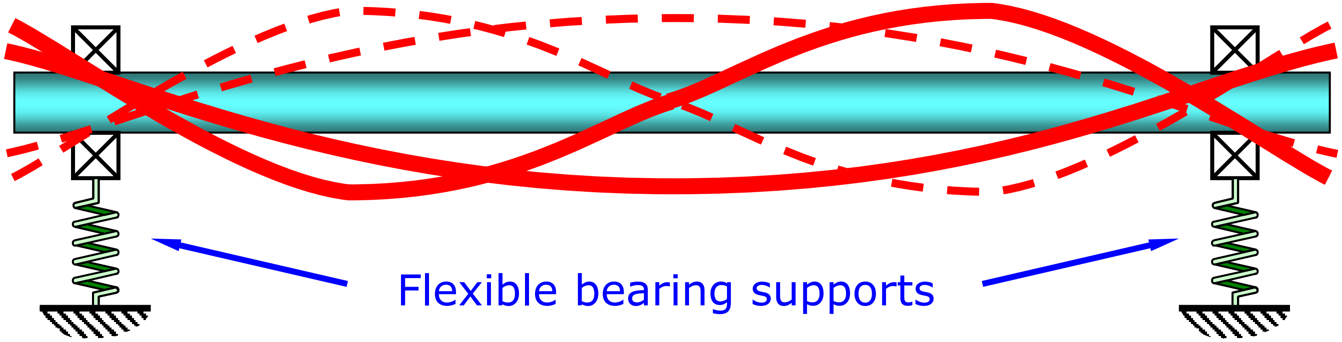
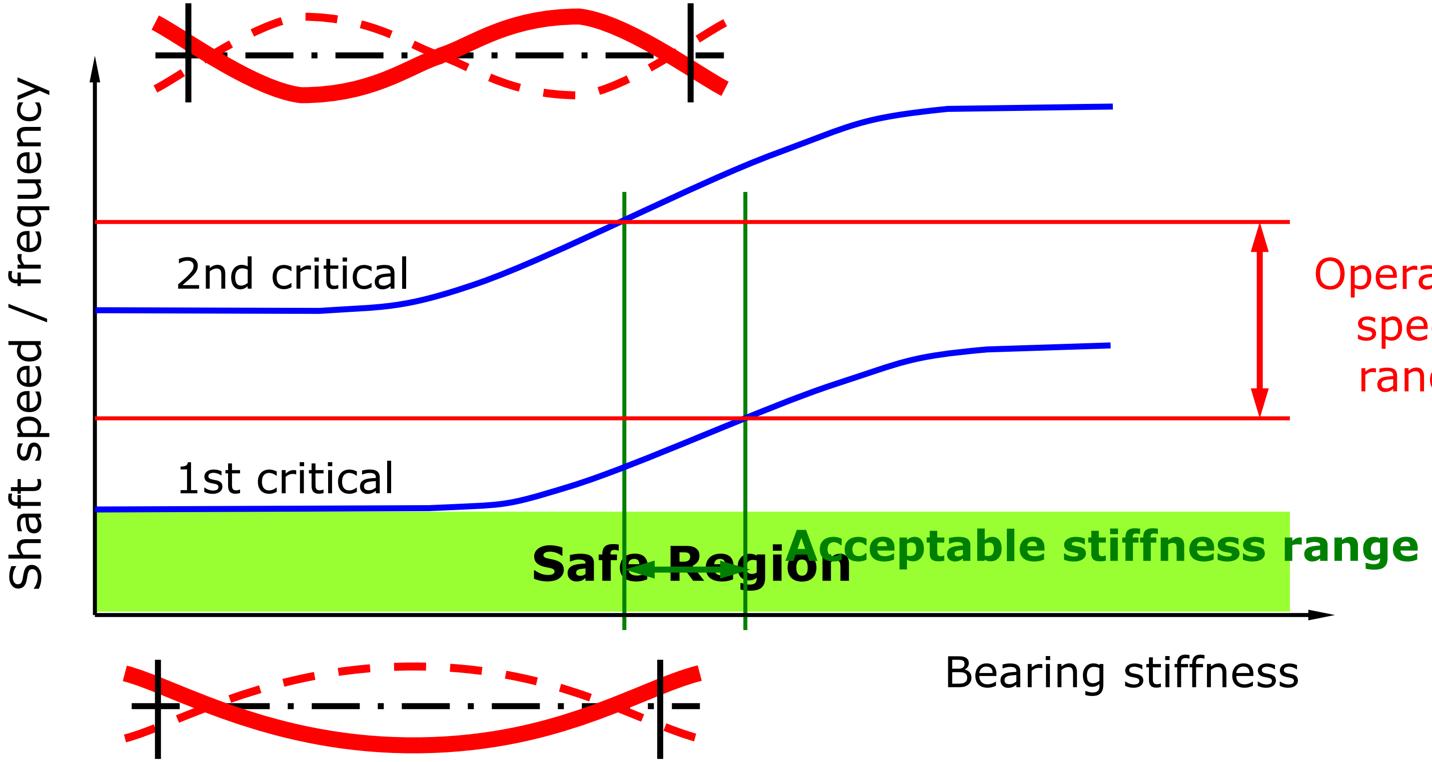
Shaft Whirl, Critical Speeds & Beam Vibration

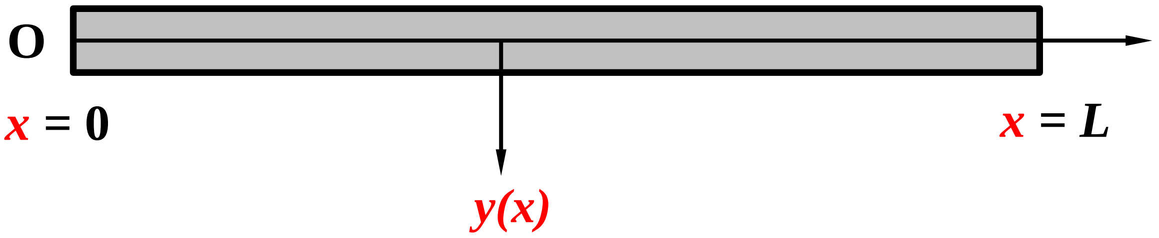


- Shaft whirl is a potentially destructive, self-sustaining flexural vibration observed in rotating shafts
- **It occurs if the rotational frequency of the shaft coincides with a resonant frequency for flexural vibration**
- These shaft speeds are called ***critical speeds***
- **If** the maximum design speed is less than the lowest critical speed, whirl will not be a problem
- This is not always possible and it is vital to be able to calculate what the critical speeds will be

Short case study – High speed drive shaft







- Given a generalized beam we wish to solve for
  - Natural Frequency ω_n or ω_{nr}
    - Where r is the frequency number (1, 2, 3, …)
  - Mode shapes associated with specific values of ω_{nr}
    - Essentially we are looking for the vertical displacement, y, for any given point along the beam, x

- From previous experience we know then that we need to find a generalized equation

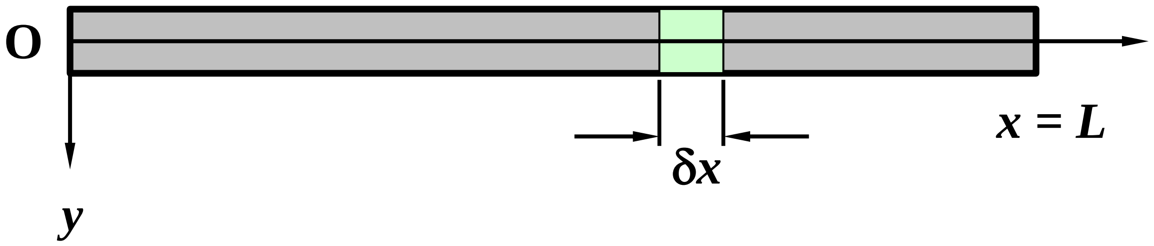
$$[Z]\{C\} = \{0\}$$

- Where $\det[Z]=0$ will give us ω_{nr}

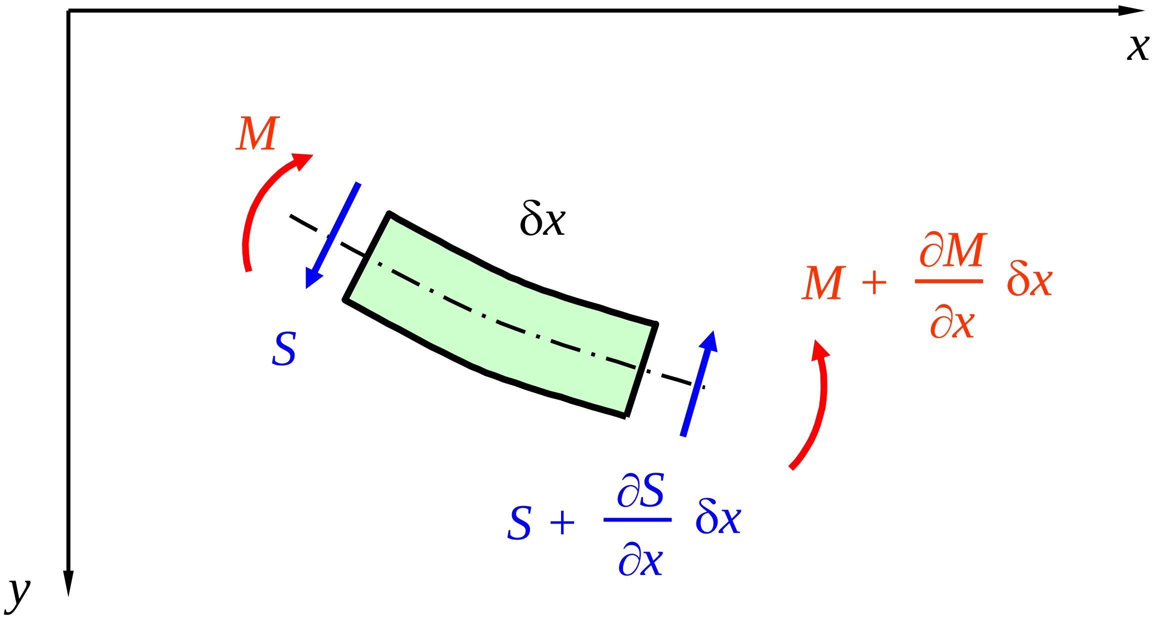
- Solving the solution vector $\{C\}$ at ω_{nr} will define the mode shapes

- To do this you need a generalized equation for vertical displacement, y, as a function of distance along the beam, x, and time, t.

Theory for the Flexural Vibration of Uniform Beams



Consider the motion of an infinitesimal element of the beam of length δx



Analysis in the handout leads to the differential equation

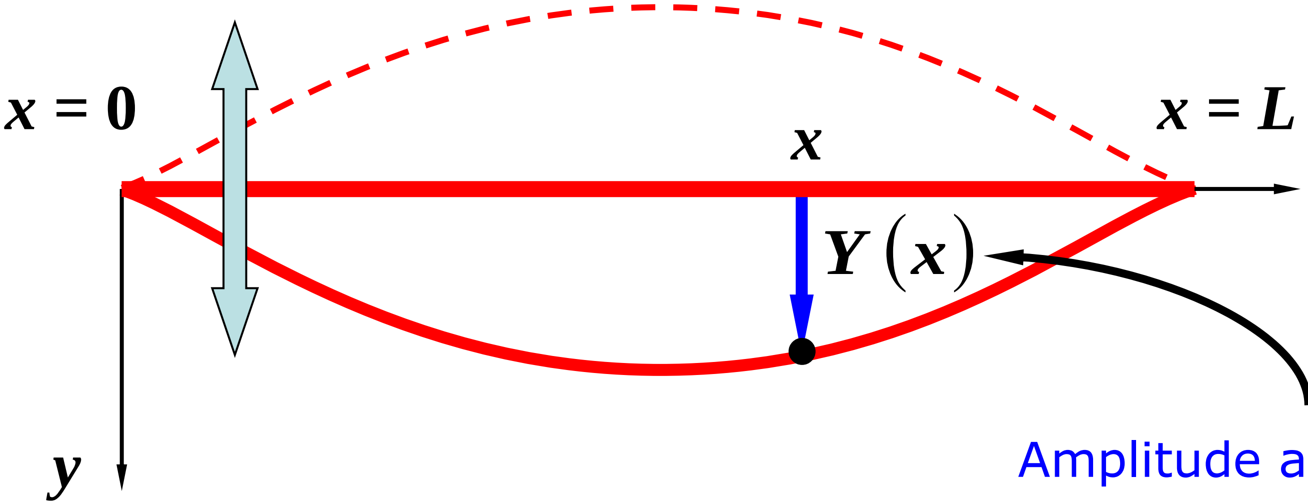
$$EI\frac{\partial^4 y}{\partial x^4} = -\rho A\frac{\partial^2 y}{\partial t^2} \qquad (4)$$

This is the general governing differential equation for the free vibration of a beam

Equation (4) is a partial differential equation giving the deflection, y, which is a function of space x and time t

We want to find the natural frequencies and the corresponding mode shapes of the beam

For free vibration at a natural frequency, **the motion of each point on the beam will be sinusoidal**, but **the amplitude of vibration will vary along the length**



Amplitude at position, x

Substitution $\boxed{y(x,t) = Y(x)\cos\omega t}$ into (4) $\boxed{EI\dfrac{\partial^4 y}{\partial x^4} = -\rho A\dfrac{\partial^2 y}{\partial t^2}}$

The deflected shape of the beam defined by the amplitude $Y(x)$ will give us the required **mode shape**

Substituting into (4), we get

$$\frac{d^4 Y}{d x^4} = \frac{\rho A \omega^2}{E I} Y(x)$$

For a uniform cross-section, ***A*** and ***I*** are constant and it's convenient to introduce the so-called ***wavenumber***, **λ**, defined by

$$\boxed{\lambda^4 = \frac{\rho A \omega^2}{E I}} \quad (5)$$

The final solution for ***Y*(*x*)** is

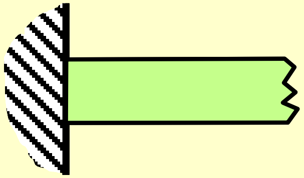
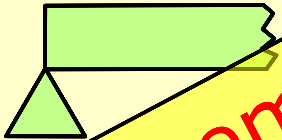
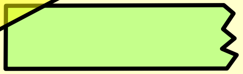
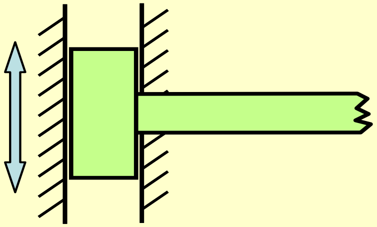
$$\boxed{Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x} \quad (6)$$

$$\boxed{Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x} \quad (6)$$

- This results in a generalized equation for displacement of y at any given point along the beam, x, for a given frequency of vibration (contained in λ)

- **HOWEVER**, this contains 4 unknowns (C_1, C_2, C_3 and C_4) and you will therefore need a minimum of 4 equations to solve for them
  - Boundary conditions must be used!!!

The constants C_1 - C_4 depend on the boundary conditions at the ends of the beam and will define the mode shapes

Descriptive terms	Diagrammatic	Boundary conditions
Built-in clamped encastré		$y = 0 \quad \frac{\partial y}{\partial x} = 0$
Simple support hinged pinned		$y = 0$ $M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$
Free		$M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$
Massless slider		$\frac{\partial y}{\partial x} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$

You will need to remember all of these

You will therefore need to partially differentiate (6)

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \quad \textbf{(6a)}$$

several times with depending on what boundary conditions you have

$$\frac{dY}{dX} = C_1 \lambda\cos\lambda x - C_2 \lambda\sin\lambda x + C_3 \lambda\cosh\lambda x + C_4 \lambda\sinh\lambda x \quad \textbf{(6b)}$$

$$\frac{d^2Y}{dx^2} = -C_1 \lambda^2\sin\lambda x - C_2 \lambda^2\cos\lambda x + C_3 \lambda^2\sinh\lambda x + C_4 \lambda^2\cosh\lambda x \quad \textbf{(6c)}$$

$$\frac{d^3Y}{dx^3} = -C_1 \lambda^3\cos\lambda x + C_2 \lambda^3\sin\lambda x + C_3 \lambda^3\cosh\lambda x + C_4 \lambda^3\sinh\lambda x \quad \textbf{(6d)}$$

General Approach for Finding the Solutions

1. Start by identifying the four boundary conditions and express the boundary conditions in terms of $Y(x)$ and its derivatives

2. Since each of the four boundary condition equations depends on C_1 - C_4, they can be assembled in the form

$$[Z]\{C\} = \{0\} \tag{7}$$

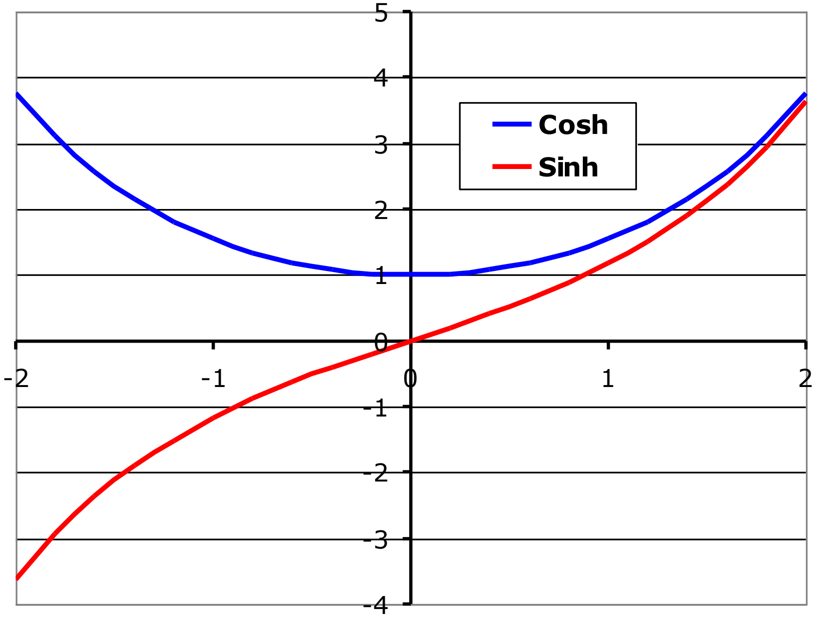
where $\{C\}$ is a vector of the constants C_1 - C_4 and $[Z]$ is a coefficient matrix.

3. For a valid solution, $\det[Z] = 0$

This gives the **Frequency Equation** and its roots will give the **natural frequencies** of the beam

4. When each root is substituted back into (7), the solution vector $\{C\}$ will define the **mode shapes** when the values are put into (6)

Example 1 Simply-supported Beam



$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sinh\theta = \cosh\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh\theta = \sinh\theta$$

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} =$$

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = -\lambda^2 C_1 \sin\lambda x - \lambda^2 C_2 \cos\lambda x + \lambda^2 C_3 \sinh\lambda x + \lambda^2 C_4 \cosh\lambda x$$

Hence, at $x = 0$, $Y = 0$ and $\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = 0$

$$Y(0) = \boxed{C_1 \times 0 + C_2 \times 1 + C_3 \times 0 + C_4 \times 1 = 0}$$

$$\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=0} = \boxed{-\lambda^2 C_1 \times 0 - \lambda^2 C_2 \times 1 + \lambda^2 C_3 \times 0 + \lambda^2 C_4 \times 1 = 0}$$

and at $x = L$, $Y = 0$ and $\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} = 0$

$$Y(L) = \boxed{C_1 \sin\lambda L + C_2 \cos\lambda L + C_3 \sinh\lambda L + C_4 \cosh\lambda L = 0}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = \boxed{-\lambda^2 C_1 \sin\lambda L - \lambda^2 C_2 \cos\lambda L + \lambda^2 C_3 \sinh\lambda L + \lambda^2 C_4 \cosh\lambda L = 0}$$

2. Assembling the four equations in matrix form

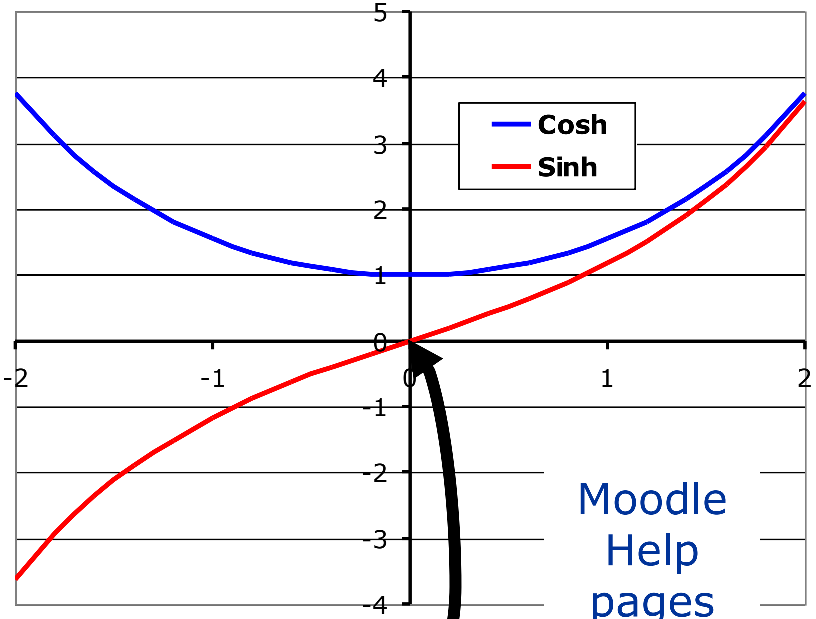
$$[Z]\{C\} = \{0\}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ \sin\lambda L & \cos\lambda L & \sinh\lambda L & \cosh\lambda L \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

3. Expanding the determinant of the coefficient matrix and equating to zero gives the **Frequency Equation.**

$$-4\lambda^4 \sin\lambda L \sinh\lambda L = 0$$

$$-4\lambda^4 \sin\lambda L \sinh\lambda L = 0$$



Moodle
Help
pages

Q1 What are the roots of the equation'

Q2 Can $\lambda = 0$?

A The definition of λ (equation 5) is

λ is only zero if the natural frequer

This is only possible if the beam ha

But a simply-supported beam does **NOT** have rigid body modes

$\therefore \lambda \neq 0$ As a result $\sinh\lambda L \neq 0$

$$\cancel{-4\lambda^4} \sin\lambda L \cancel{\sinh\lambda L} = 0 \Rightarrow \boxed{\sin\lambda L = 0}$$

The Frequency Equation is $\sin\lambda L = 0$

which has roots $\lambda_r L = r\pi$ for $\boldsymbol{r} \mathbf{= 1, 2, 3, \ldots}$

Since $\lambda^4 = \dfrac{\rho A \omega^2}{E I}$ the **natural frequencies** are

$$\omega_r = \omega_{nr} = \left(\frac{r\pi}{L}\right)^2 \sqrt{\frac{E I}{\rho A}} \quad \text{for} \quad \boldsymbol{r} \mathbf{= 1, 2, 3, \ldots}$$

2. Assemble into matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & \text{-}\lambda & 0 & \lambda \\ \sin\lambda L & \cos\lambda L & sinh\lambda L & \cosh\lambda L \\ -\lambda^2 \sin\lambda L & -\lambda^2 \cos\lambda L & \lambda^2 \sinh\lambda L & \lambda^2 \cosh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (7a) \\ (7b) \\ (7a) \\ (7c) \end{matrix}$$

$$[Z]\{C\} = \{0\}$$

3. Solving $\det[Z]=0$ gives the **Frequency Equation** and its roots will give ω_r contained in λ_r

- This is complicated so we have given you the resulting Frequency Equation for a number of different beam types on **page 5** of your notes

- But this is still difficult to solve, so we also give you the numerical solutions for $\lambda_r L$ on the same page

Numerical values of roots $\lambda_r L$ of frequency equations

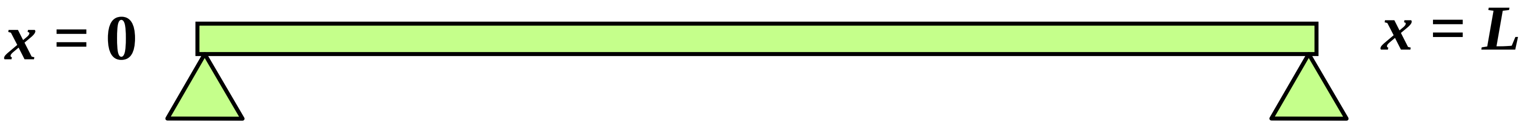
r	**1**	**2**	**3**	**4**	**5**	**>5**
Pinned-pinned	π	**2** π	**3** π	**4** π	**5** π	***r*** π
Clamped-clamped & free-free	**4.730**	**7.853**	**10.996**	**14.137**	**17.279**	**≈ (*r* + 0.5)** π
Clamped-pinned & free-pinned	**3.927**	**7.069**	**10.210**	**13.351**	**16.493**	**≈ (*r* + 0.25)** π
Clamped-free	**1.875**	**4.694**	**7.855**	**10.996**	**14.137**	**≈ (*r* – 0.5)** π

Selecting the values of $\lambda_r L$ from the above table for the beam of interest, the natural frequencies can be found from reworking equation (5). That is:

$$\omega_r = \omega_{nr} = \frac{(\lambda_r L)^2}{L^2}\sqrt{\frac{E\,I}{\rho\,A}}$$

where $\omega_{n1} = \frac{(\pi)^2}{L^2}\sqrt{\frac{E\,I}{\rho\,A}}$ $\omega_{n2} = \frac{(2\pi)^2}{L^2}\sqrt{\frac{E\,I}{\rho\,A}}$,etc.

Example 1 Simply-supported Beam



The four boundary conditions lead to

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ \sin\lambda L & \cos\lambda L & \sinh\lambda L & \cosh\lambda L \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (7)$$

The frequency equation is $\det[Z] = 0$

which has roots $\lambda_r L = r\pi$ for $\boldsymbol{r} = 1, 2, 3, \ldots$ (from previous table)

so the **natural frequencies** are

$$\omega_r = \left(\frac{r\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \quad \text{for } \boldsymbol{r} = 1, 2, 3, \ldots$$

4. To find the **mode shapes**, substitute the roots into equation (7) and solve for the constants $\boldsymbol{C_1}$ **-** $\boldsymbol{C_4}$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda_r^2 & 0 & \lambda_r^2 \\ \sin\lambda_r L & \cos\lambda_r L & \sinh\lambda_r L & \cosh\lambda_r L \\ -\lambda_r^2\sin\lambda_r L & -\lambda_r^2\cos\lambda_r L & \lambda_r^2\sinh\lambda_r L & \lambda_r^2\cosh\lambda_r L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (7a) \\ (7b) \\ (7c) \\ (7d) \end{matrix}$$

(7a) ⇒ $C_2 + C_4 = 0$

(7b) ⇒ $\lambda_r^2\left(-C_2 + C_4\right) = 0$

Since $\lambda_r \neq 0$ ⇒ $\boxed{C_2 = C_4 = 0}$

(7c) ⇒ $\sin\lambda_r L . C_1 + \sinh\lambda_r L . C_3 = 0$

(7d) ⇒ $-\lambda_r{}^2 \sin\lambda_r L . C_1 + \lambda_r{}^2 \sinh\lambda_r L . C_3 = 0$

⇒ $\boxed{\therefore\ C_3 = 0}$

The only non-zero constant is $\boldsymbol{C_1}$

Its value is arbitrary & we normally choose $\boldsymbol{C_1 = 1}$

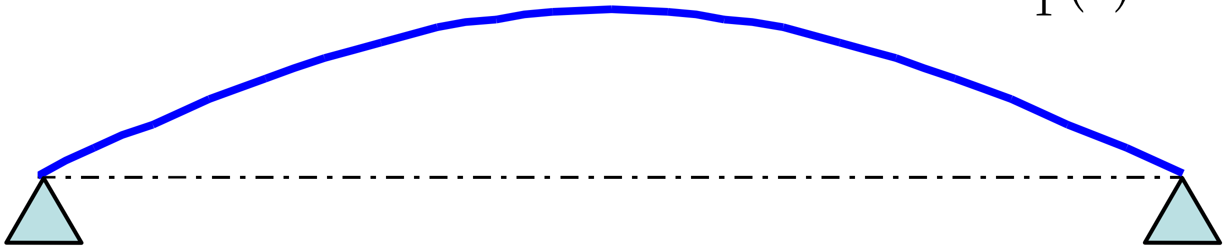
$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \qquad (6)$$

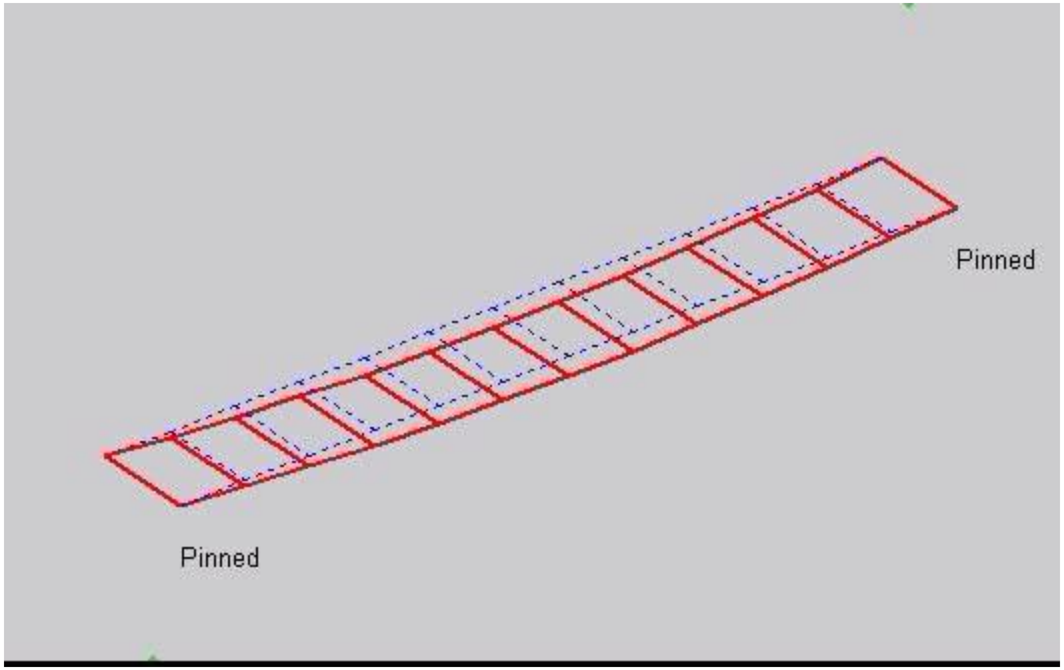
Hence, the mode shape is

$$\boxed{Y_r(x) = \sin\lambda_r x = \sin\frac{r\pi x}{L}}$$

Mode #1 $(r = 1)$

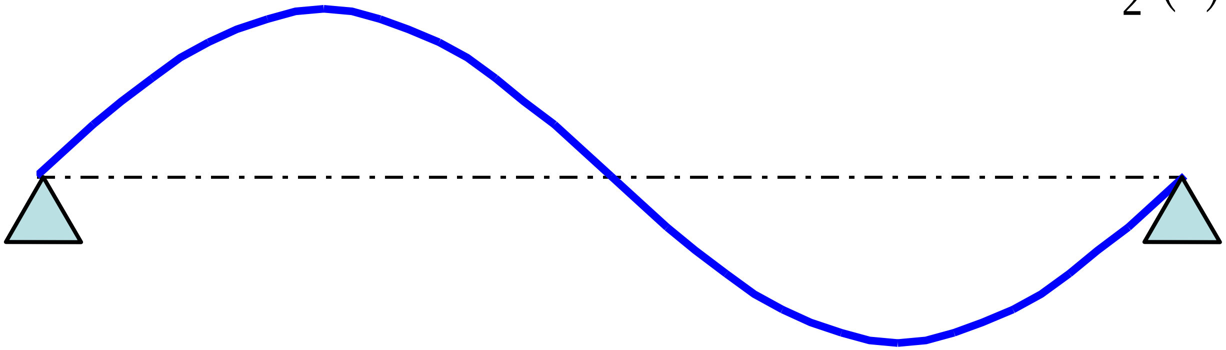
$$Y_1(x) = \sin\frac{\pi x}{L}$$

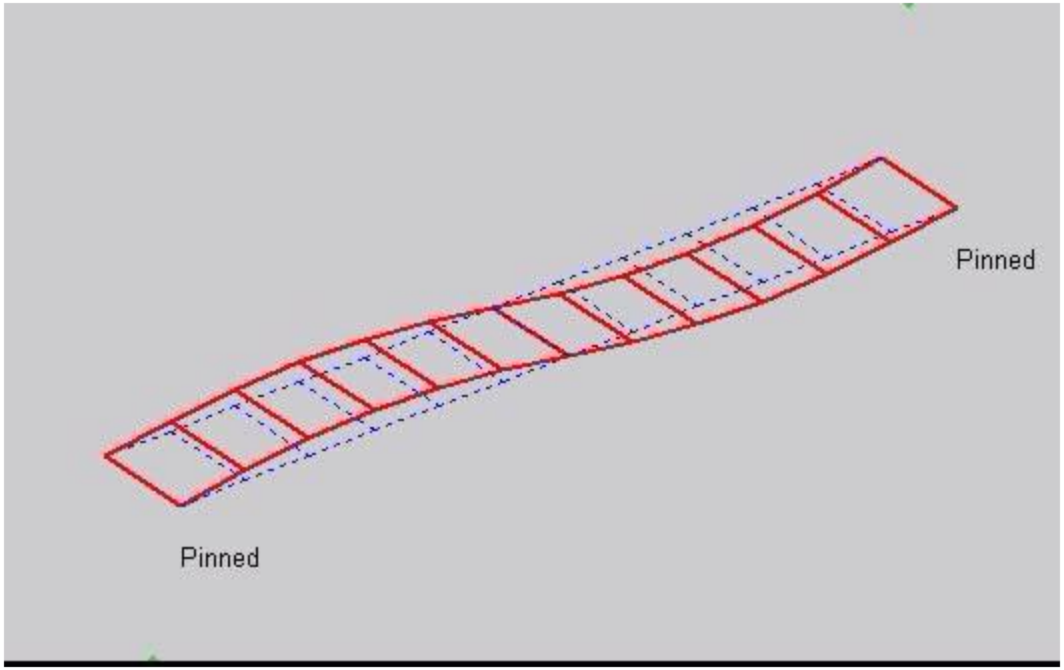




Mode #2 (*r* = 2)

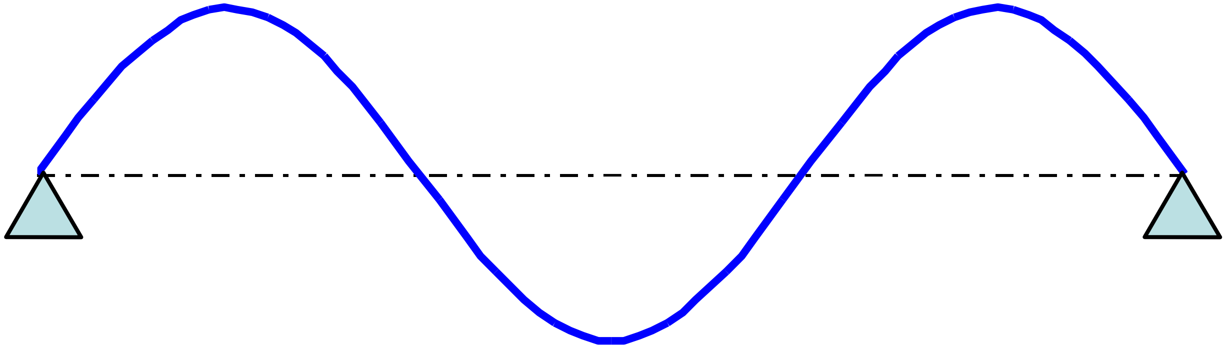
$$Y_2(x) = \sin\frac{2\pi x}{L}$$

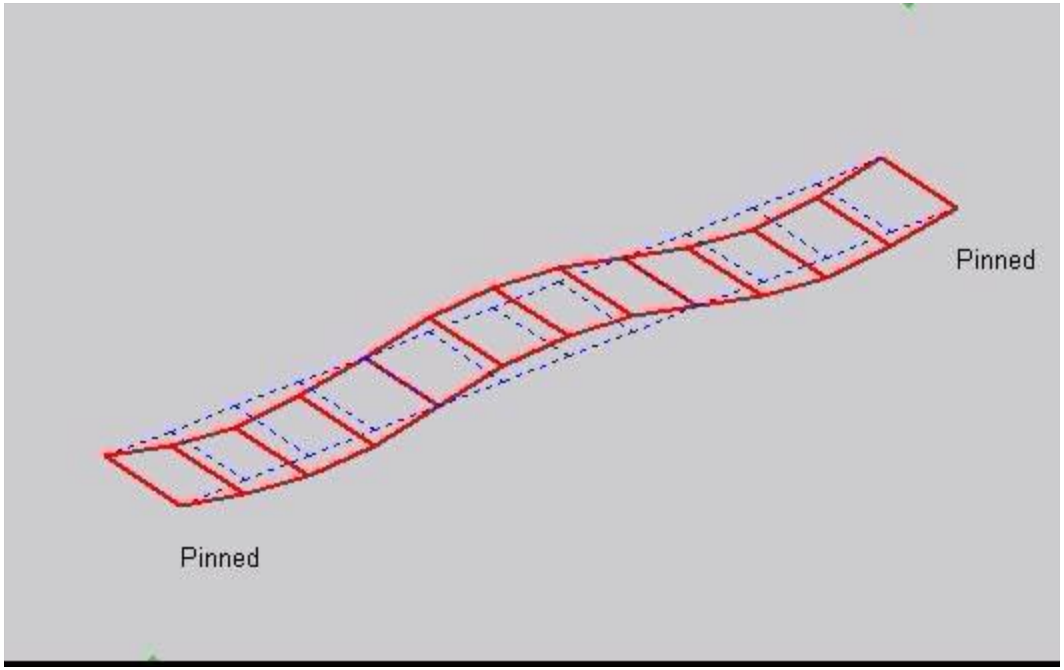




Mode #3 (*r* = 3)

$$Y_3(x) = \sin\frac{3\pi x}{L}$$





Example 2   Cantilever (Clamped-free) Beam

1. Boundary conditions

The boundary conditions are

Clamped end at $x = 0, \;\; y = 0$ and $\dfrac{\partial y}{\partial x} = 0$

Free end at $x = L, \;\; M = 0 \;\; \therefore \dfrac{\partial^2 y}{\partial x^2} = 0$

and $S = 0 \;\; \therefore \dfrac{\partial^3 y}{\partial x^3} = 0$

Since $y(x,t) = Y(x)\cos\omega t$ the boundary conditions become

$$\text{At } x = 0\ , \quad \boxed{Y = 0} \quad \text{and} \quad \boxed{\frac{\mathrm{d}Y}{\mathrm{d}x} = 0}$$

$$\text{At } x = L, \quad \boxed{\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = 0} \quad \text{and} \quad \boxed{\frac{\mathrm{d}^3Y}{\mathrm{d}x^3} = 0}$$

2. Assemble into matrix form

Substituting from equation (6a, 6b, 6c and 6d) we get (in matrix form)

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \\ -\lambda^3\cos\lambda L & \lambda^3\sin\lambda L & \lambda^3\cosh\lambda L & \lambda^3\sinh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} \text{(7a)} \\ \text{(7b)} \\ \text{(7c)} \\ \text{(7d)} \end{matrix}$$

This is the particular version of equation (7) for a cantilever beam

3. **Set up the Frequency Equation**

The **Frequency Equation** is given by $\det[Z]=0$

After manipulation (and noting that a cantilever has no rigid body modes), this gives

$$1+\cos\lambda L\cosh\lambda L = 0$$

There are no closed-form solutions to this equation, so the roots $\lambda_r L$ must be obtained numerically and are given in the handout on **page 5**

Numerical values of roots $\lambda_r L$ of frequency equations

r	1	2	3	4	5	>5
Pinned-pinned	π	2π	3π	4π	5π	$r\pi$
Clamped-clamped & free-free	4.730	7.853	10.996	14.137	17.279	$\approx (r + 0.5)\pi$
Clamped-pinned & free-pinned	3.927	7.069	10.210	13.351	16.493	$\approx (r + 0.25)\pi$
Clamped-free	1.875	4.694	7.855	10.996	14.137	$\approx (r - 0.5)\pi$

Selecting the values of $\lambda_r L$ from the above table for the beam of interest, the natural frequencies can be found from equation (5). That is:

$$\omega_r = \omega_{nr} = \frac{(\lambda_r L)^2}{L^2} \sqrt{\frac{E\,I}{\rho\,A}}$$

where $\omega_{n1} = \frac{(1.875)^2}{L^2} \sqrt{\frac{E\,I}{\rho\,A}}$ $\quad \omega_{n2} = \frac{(4.694)^2}{L^2} \sqrt{\frac{E\,I}{\rho\,A}}$,etc.

4. Find the Modes Shapes

The **mode shapes** are obtained by substituting $\lambda = \lambda_r$ into equation (7) and solving for the constants C_1 to C_4

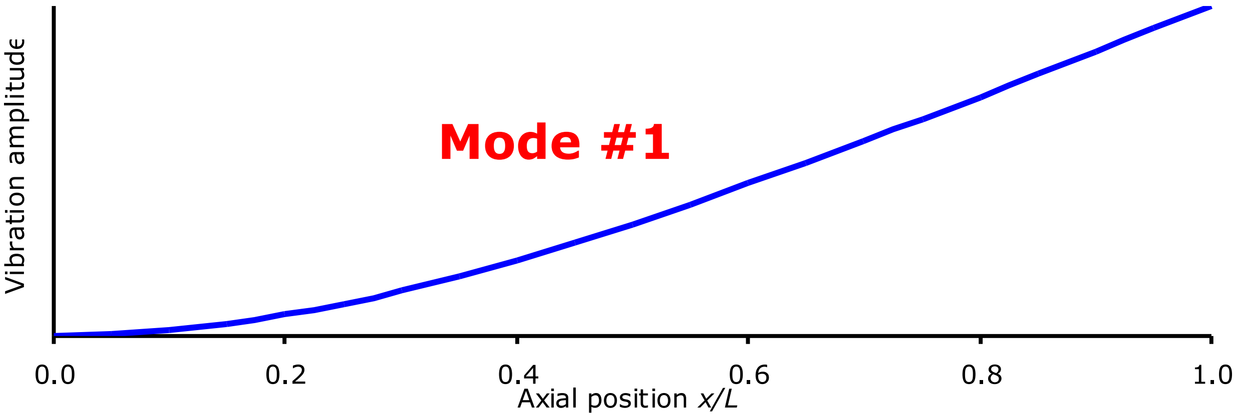
From (7a) and (7b) $C_3 = -C_1$ and $C_4 = -C_2$

Thus from (7c) or (7d)

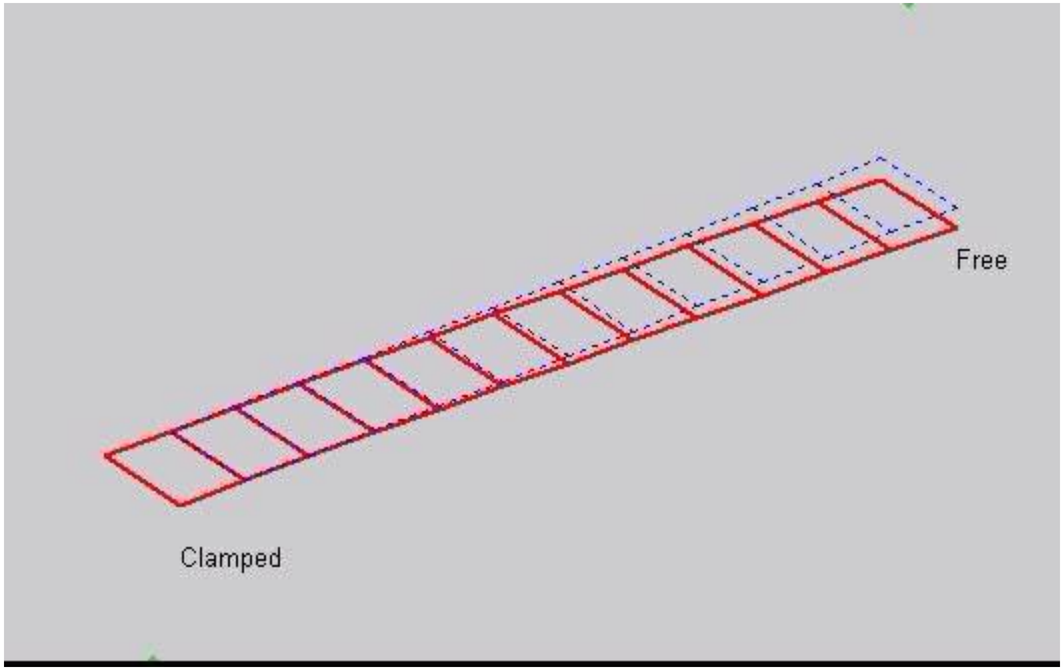
$$C_2 = \boxed{-\frac{\sin \lambda_r L + \sinh \lambda_r L}{\cos \lambda_r L + \cosh \lambda_r L}} C_1 = \boxed{\sigma_r} C_1$$

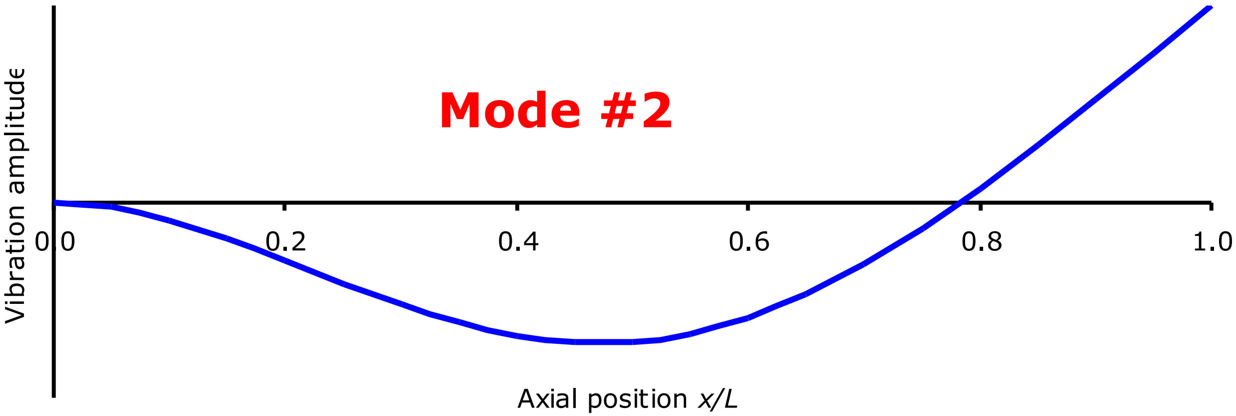
If we choose $C_1 = 1$, the mode shape becomes

$$\boxed{Y_r(x) = \sin \lambda_r x - \sinh \lambda_r x + \sigma_r \left(\cos \lambda_r x - \cosh \lambda_r x\right)}$$

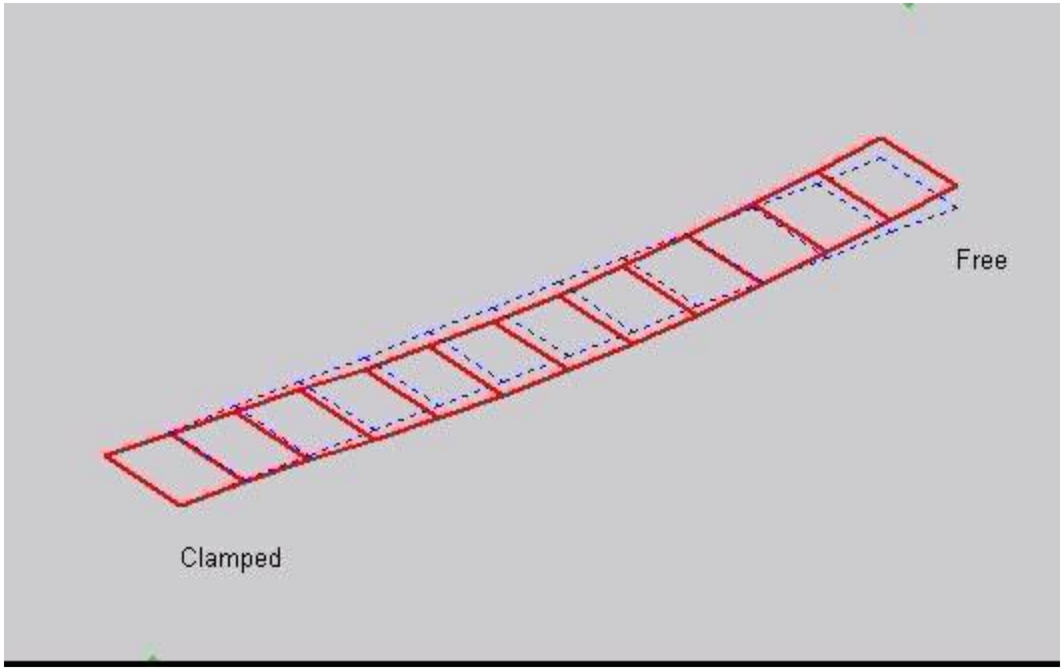


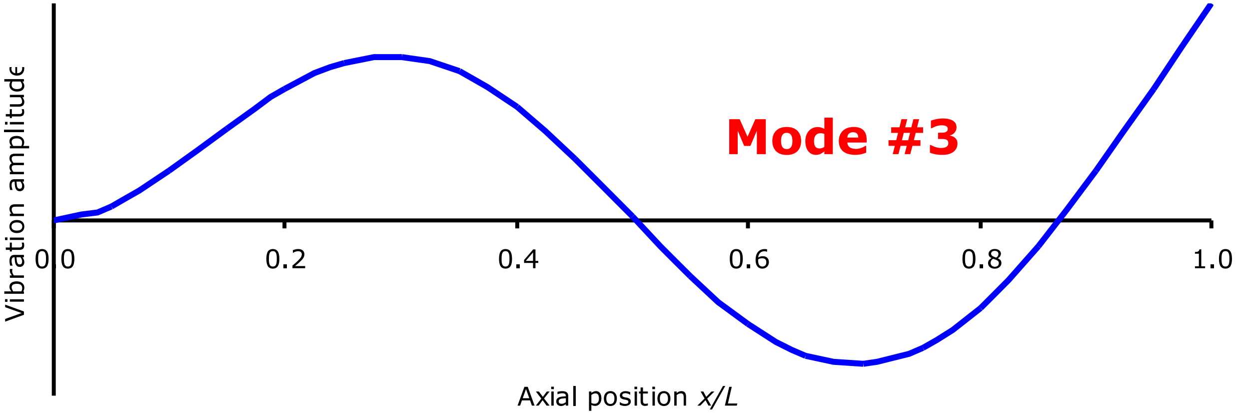
$$Y_r(x) = \sin\frac{1.875x}{L} - \sinh\frac{1.875x}{L} + \sigma_r\left(\cos\frac{1.875x}{L} - \cosh\frac{1.875x}{L}\right)$$



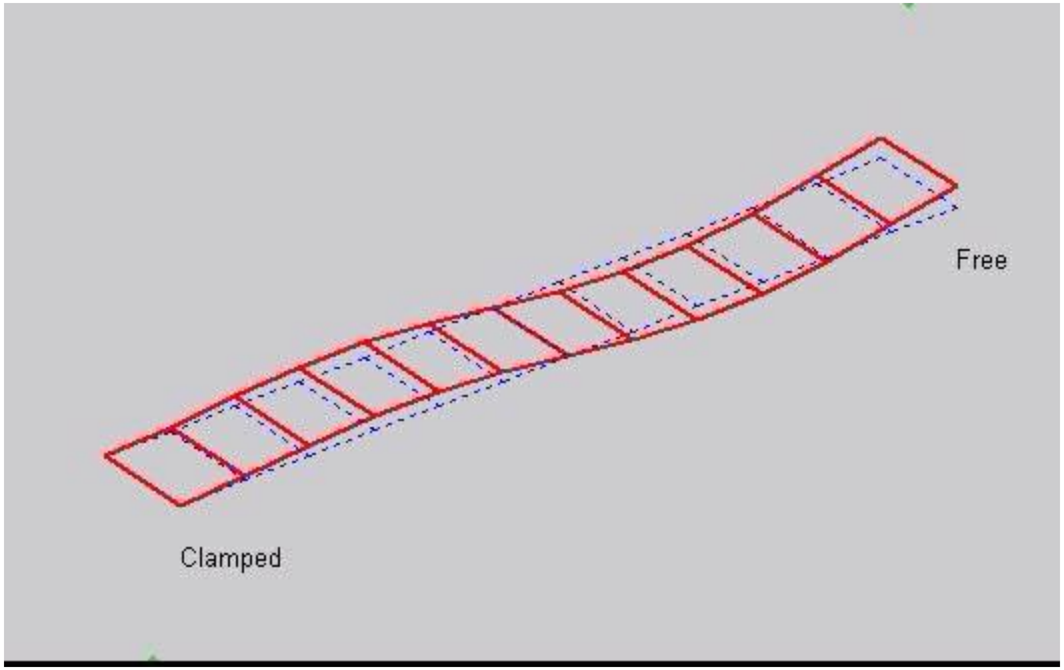


$$Y_r(x) = \sin\frac{4.694\,x}{L} - \sinh\frac{4.694\,x}{L} + \sigma_r\left(\cos\frac{4.694\,x}{L} - \cosh\frac{4.694\,x}{L}\right)$$

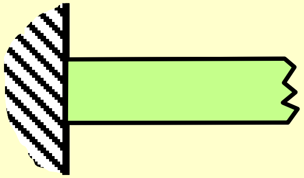
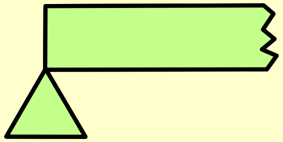
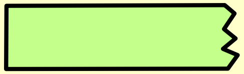
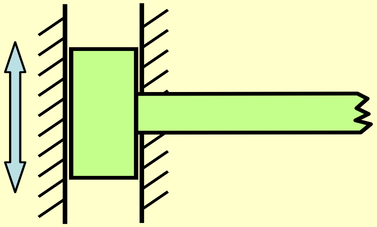




$$Y_r(x) = \sin\frac{7.855x}{L} - \sinh\frac{7.855x}{L} + \sigma_r\left(\cos\frac{7.855x}{L} - \cosh\frac{7.855x}{L}\right)$$



“Standard” boundary conditions

Descriptive terms	Diagrammatic	Boundary conditions
Built-in clamped encastré		$y = 0 \quad \frac{\partial y}{\partial x} = 0$
Simple support hinged pinned		$y = 0$ $M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$
Free		$M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$
Massless slider		$\frac{\partial y}{\partial x} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$

Other Boundary Conditions

Example Cantilever Beam with a Mass at the Free End

Mass m

M of I I_M

1. Boundary conditions

Clamped end at $x = 0,\ \ y = 0$ and $\dfrac{\partial y}{\partial x} = 0$

so $\boxed{Y = 0}$ and $\boxed{\dfrac{\mathrm{d}Y}{\mathrm{d}x} = 0}$ as before

However, at $x = L,\ \boxed{S \neq 0}$ and $\boxed{M \neq 0}$

Apply the principles of

1. Compatibility of displacements

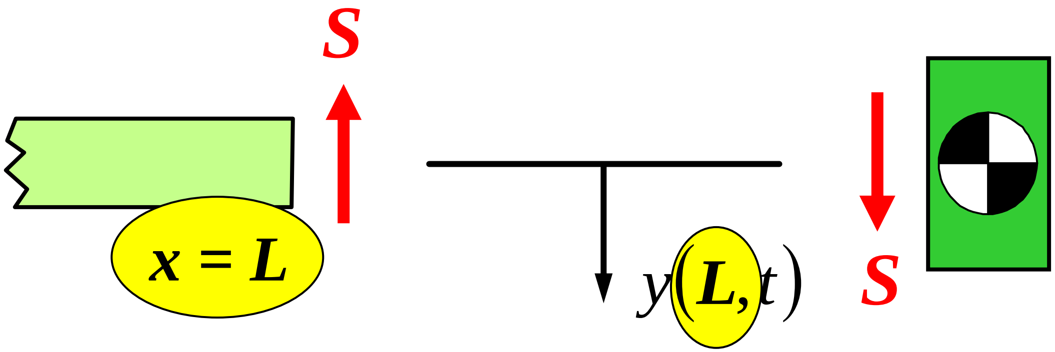
2. Equilibrium of forces and moments

Apply the principles of

1. **Compatibility of displacements**
2. **Equilibrium of forces**

Consider the shear force reaction between the beam and the mass

Free Body Diagram (separate the mass from the beam)

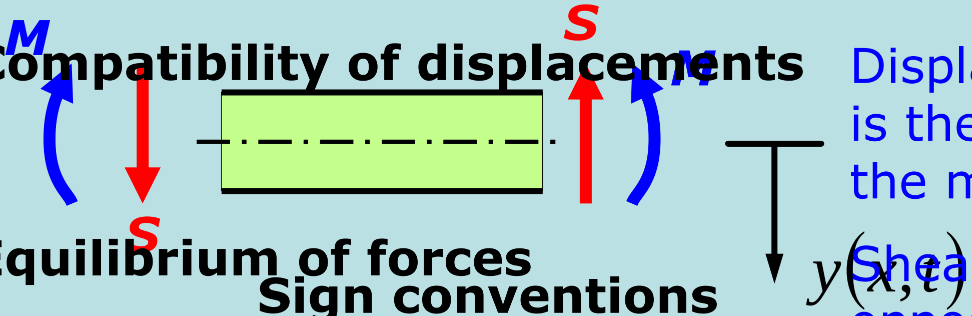


Compatibility of displacements

Displacement at the end of the beam is the same as the displacement of the mass

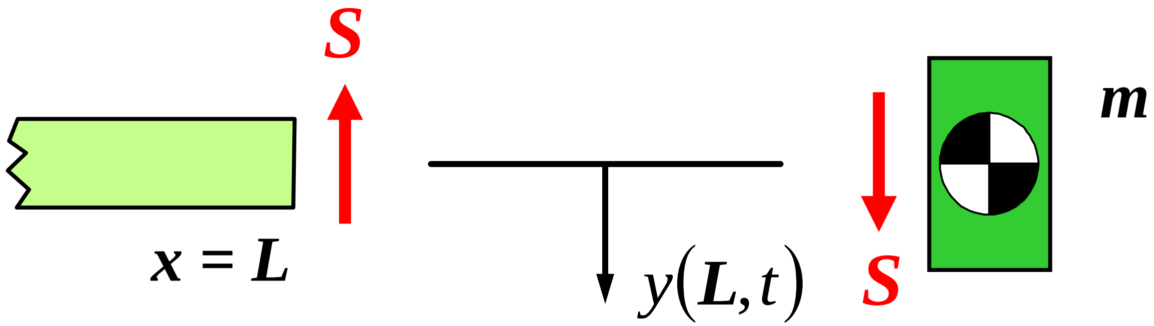
Equilibrium of forces

Shear force on the beam is equal and opposite to the force on the mass



Sign conventions

See Formula Sheet



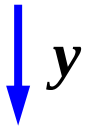
For the beam

$$S(t) = EI\left(\frac{\partial^3 y}{\partial x^3}\right)_{x=L}$$

But $y(x,t) = Y(x)\cos\omega t$

$$\boxed{S(t) = EI\left(\frac{\mathrm{d}^3 Y}{\mathrm{d}x^3}\right)_{x=L}\cos\omega t}$$

For the mass



$$S(t) = m\left(\frac{\partial^2 y}{\partial t^2}\right)_{x=L}$$

$$\boxed{S(t) = m\times\left(-\omega^2\,Y(L)\cos\omega t\right)}$$

Equating and noting that

$$\omega^2 = \frac{EI\lambda^4}{\rho A}$$

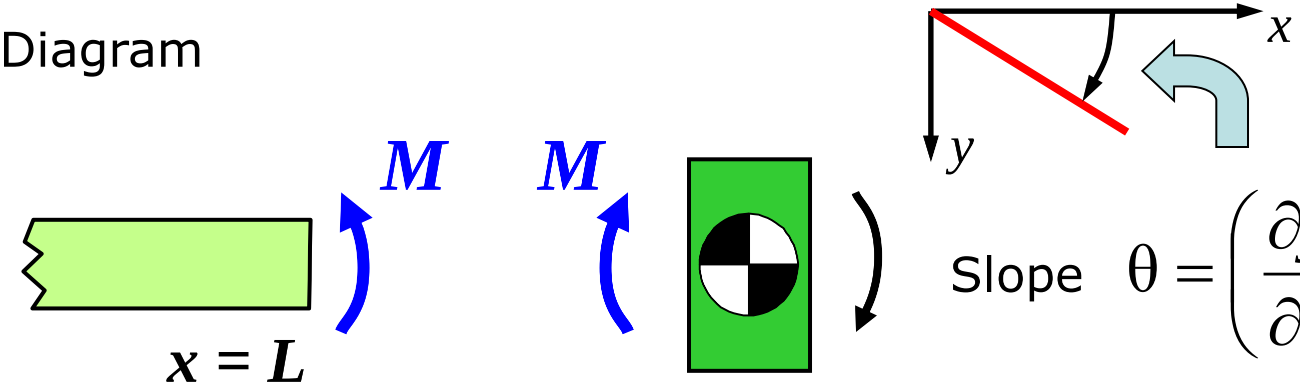
$$\boxed{\left(\frac{\mathrm{d}^3 Y}{\mathrm{d}x^3}\right)_{x=L} + \frac{m(\lambda L)^4}{\rho A L^4}\,Y(L) = 0}$$

Apply the principles of

1. **Compatibility of displacements**
2. **Equilibrium of moments**

Consider the bending moment reaction between the beam and the mass

Free Body Diagram



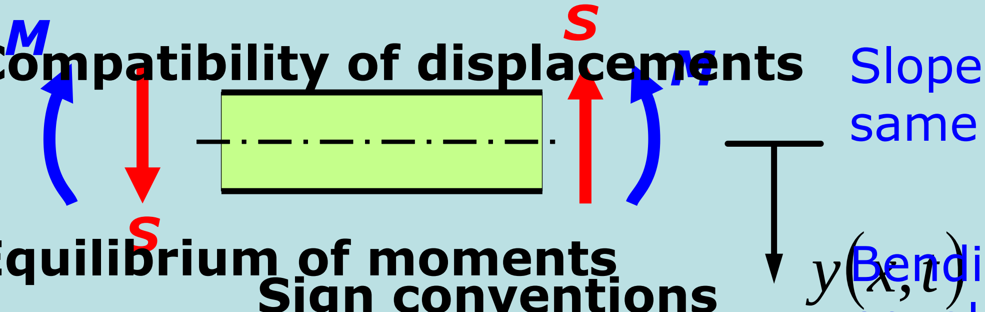
Slope $\theta = \left(\frac{\partial y}{\partial x}\right)_{x=L}$

Compatibility of displacements

Slope at the end of the beam is the same as the rotation of the mass

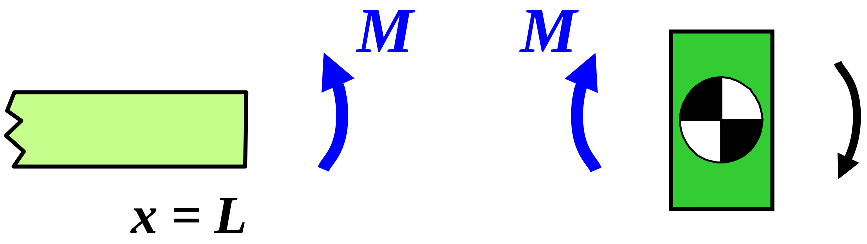
Equilibrium of moments

Bending moment on the beam is equal and opposite to the bending moment on the mass



Sign conventions

See Formula Sheet



Slope $\theta = \left(\frac{\partial y}{\partial x}\right)_{x=L}$

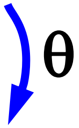
For the beam

$$M(t) = -EI\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=L}$$

But $y(x,t) = Y(x)\cos\omega t$

$$M(t) = -EI\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L}\cos\omega t$$

For the mass



$$M(t) = I_{\mathrm{M}}\left(\frac{\partial^2\theta}{\partial t^2}\right)_{x=L}$$

$$\theta(t) = \frac{\partial y}{\partial x} = \left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)\cos\omega t$$

$$\frac{\partial^2\theta}{\partial t^2} = -\omega^2\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)\cos\omega t$$

$$M(t) = -I_{\mathrm{M}}\,\omega^2\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=L}\cos\omega t$$

Equating

$$\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L} - \frac{I_{\mathrm{M}}(\lambda L)^4}{\rho A L^4}\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=L} = 0$$

Collecting the boundary condition equations together

$$Y(0) = 0$$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=0} = 0$$

$$\left(\frac{\mathrm{d}^3Y}{\mathrm{d}x^3}\right)_{x=L} + \frac{m(\lambda L)^4}{\rho A L^4} Y(L) = 0$$

$$\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L} - \frac{I_{\mathrm{M}}(\lambda L)^4}{\rho A L^4} \left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=L} = 0$$

2. Assemble into matrix form

Substitute for $Y(x)$ and its derivatives to give the new equation (7)

Steps 3 and 4 follow as in the previous examples